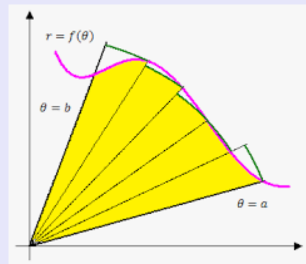
Calculus II

Lecture 16



Feb 19-8:47 AM

Class QZ 13

Use $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ to find

the arc length of curve given by

$x = 5\cos t$ and $y = 5\sin t$ for $0 \le t \le 2\pi$.

$\frac{dx}{dt} = -5\sin t$ $\quad \frac{dy}{dt} = 5\cos t$ $\quad L = \int_0^{2\pi} \sqrt{(-5\sin t)^2 + (5\cos t)^2}\, dt$

$L = \int_0^{2\pi} \sqrt{25(\sin^2 t + \cos^2 t)}\, d = \int_0^{2\pi} \sqrt{25 \cdot 1}\, dt$

$\sin t = \frac{y}{5}$, $\cos t = \frac{x}{5}$ $\quad = \int_0^{2\pi} 5\, dt = 5t \Big|_0^{2\pi}$

$\sin^2 t + \cos^2 t = 1$

$\frac{y^2}{25} + \frac{x^2}{25} = 1 \rightarrow x^2 + y^2 = 25$ $\quad = \boxed{10\pi}$

Circle

Center $(0,0)$

radius 5

Circumf.

$C = 2\pi r$

$= 2\pi \cdot 5$

$= \boxed{10\pi}$

Jul 8-10:53 AM

$x = \frac{1}{2}\cos\theta$, $y = 2\sin\theta$ $\boxed{0 \le \theta \le \pi}$

QI & QII

$y \ge 0$

1) Find eqn in x & y only.

$\cos\theta = 2x$

$\sin\theta = \frac{y}{2}$

$(2x)^2 + \left(\frac{y}{2}\right)^2 = 1$

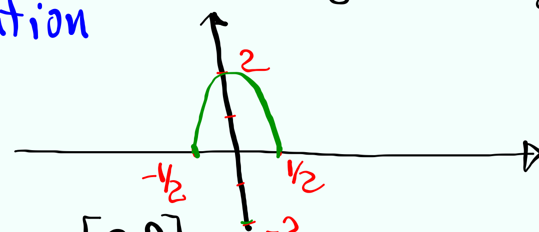
$4x^2 + \frac{y^2}{4} = 1$

Ellipse

$x = 0 \rightarrow y = \pm 2$

$y = 0 \rightarrow x = \pm\frac{1}{2}$

2) Draw the equation

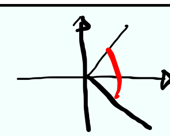


$[-\frac{1}{2}, \frac{1}{2}]$ $[0, 2]$

3) Discuss domain & Range.

Jul 9-8:18 AM

$x = \tan^2\theta$, $y = \sec\theta$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$



$x \ge 0$ $y \ge 1$

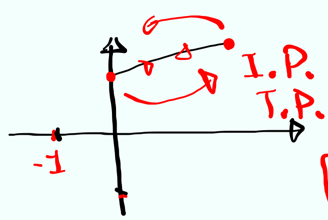
1) Find eqn in x & y

$1 + \tan^2\theta = \sec^2\theta$

$1 + x = y^2$

$x = y^2 - 1$

2) Draw



$\theta = -\frac{\pi}{4}$ $x = 1$ $y = \sqrt{2}$

$\theta = \frac{\pi}{4}$ $x = 1$ $y = \sqrt{2}$

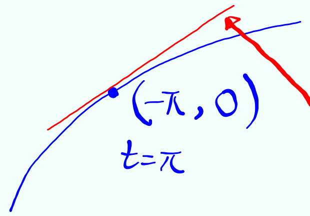
3) Discuss domain & Range.

$\theta = 0$ $x = 0$ $y = 1$

Jul 9-8:23 AM

find eqn of tan. line to the curve given by

$x = t\cos t$, $y = t\sin t$ at $t = \pi$.



$x = \pi\cos\pi = -\pi$

$y = \pi\sin\pi = 0$

$m = \left.\dfrac{dy}{dx}\right|_{\substack{(-\pi,0)\\ t=\pi}}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1\cdot\sin t + t\cdot\cos t}{1\cdot\cos t + t\cdot -\sin t}$$

$$m = \frac{\cancel{\sin\pi}^{\,0} + \pi\cancel{\cos\pi}^{\,-1}}{\cancel{\cos\pi}^{\,-1} - \pi\cancel{\sin\pi}^{\,0}} = \pi$$

$$y - 0 = \pi(x - -\pi)$$

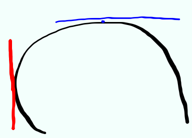
$$\boxed{y = \pi(x + \pi)}$$

Jul 9-8:36 AM

find points on the graph of

$x = t^3 - 3t$, $y = t^3 - 3t^2$ where the

tan. line is horizontal or Vertical.



Horizontal tan. line $\rightarrow m = 0$

Vertical tangent line
m is undefined.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 6t}{3t^2 - 3}$$

Horizontal tan. line

$\dfrac{dy}{dt} = 0$ & $\dfrac{dx}{dt} \neq 0$

$3t^2 - 6t = 0$ $\quad 3t(t-2) = 0$

$t = 0 \rightarrow (0, 0)$

$t = 2 \rightarrow (2, -4)$

$\dfrac{dx}{dt} = 0$ & $\dfrac{dy}{dt} \neq 0$

$3t^2 - 3 = 0$ $\quad 3(t^2 - 1) = 0$

$t = \pm 1$

$t = 1 \rightarrow (-2, -2)$

$t = -1 \rightarrow (2, -4)$

Jul 9-8:42 AM

Discuss concavity of the curve given by

$x = t^2+1$ & $y = e^t - 1$

Concave up $f''>0$

Concave down $f''<0$

1) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{e^t}{2t}$

2) $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}\left[\frac{e^t}{2t}\right]}{2t}$

$= \dfrac{\frac{e^t \cdot 2t - e^t \cdot 2}{(2t)^2}}{2t} = \boxed{\dfrac{2e^t(t-1)}{8t^3}}$

$\dfrac{d^2y}{dx^2} = \dfrac{e^t(t-1)}{4t^3}$ $\quad e^t > 0$

$t-1=0 \rightarrow t=1$

$4t^3=0 \rightarrow t=0$

$t=-1$		$t=\frac{1}{2}$		$t=2$
$f''>0$	0	$f''<0$	1	$f''>0$
C.U.		C.D.		C.U.

Inflection pts at $t=0$, and $t=1$

$t=0 \rightarrow (1,0)$

$t=1 \rightarrow (2, e-1)$

Jul 9-8:55 AM

Find the exact length of the curve given by $x = t\sin t$, $y = t\cos t$ for $0 \le t \le 1$.

$L = \displaystyle\int_0^1 \sqrt{\underbrace{(\sin t + t\cos t)^2}_{(A+B)^2} + \underbrace{(\cos t - t\sin t)^2}_{(A-B)^2}}\, dt$

$\dfrac{dx}{dt} = \sin t + t\cos t$

$\dfrac{dy}{dt} = \cos t - t\sin t$

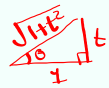
$= \displaystyle\int_0^1 \sqrt{\sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t + \cos^2 t - \cancel{2t\sin t\cos t} + t^2\sin^2 t}\; dt$

$= \displaystyle\int_0^1 \sqrt{1+t^2}\, dt$

$t = \tan\theta$

$\sqrt{1+t^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$

$dt = \sec^2\theta\, d\theta$



$= \displaystyle\int \sec\theta \cdot \sec^2\theta\, d\theta$

$= \displaystyle\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$

$= \dfrac{1}{2}\cdot\sqrt{1+t^2}\cdot t + \dfrac{1}{2}\ln\left|\sqrt{1+t^2} + t\right| \Big|_0^1$

$= \dfrac{1}{2}\left[t\sqrt{1+t^2} + \ln\left|\sqrt{1+t^2} + t\right|\right]\Big|_0^1 = \boxed{\dfrac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right]}$

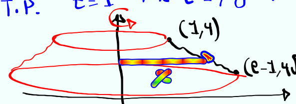
Jul 9-9:09 AM

Find the surface area by rotating the given curve $x = e^t - t$, $y = 4e^{t/2}$ for $0 \le t \le 1$ about y-axis.

$\frac{dx}{dt} = e^t - 1$ $\frac{dy}{dt} = 4 \cdot e^{t/2} \cdot \frac{1}{2} = 2e^{t/2}$

I.P. $t=0 \to x=1, y=4 \to (1,4)$

T.P. $t=1 \to x=e-1, y=4e^{1/2} \to (e-1, 4\sqrt{e})$



$$S = \int_0^1 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 2\pi (e^t - t) \cdot \sqrt{(e^t-1)^2 + (2e^{t/2})^2}\, dt$$

$$= 2\pi \int_0^1 (e^t - t) \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$= 2\pi \int_0^1 (e^t - t) \sqrt{e^{2t} + 2e^t + 1}\, dt$$

$$= 2\pi \int_0^1 (e^t - t) \sqrt{(e^t + 1)^2}\, dt$$

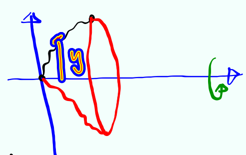
$$= 2\pi \int_0^1 (e^t - t)(e^t + 1)\, dt$$

foil, do integration, evaluate

Jul 9-9:24 AM

Find the surface area by rotating the curve below by x-axis.

$x = 3t - t^3$, $y = 3t^2$, $0 \le t \le 1$



I.P. $t=0$ $x=0$, $y=0$

T.P. $t=1$ $x=2$, $y=3$

$$S = \int_0^1 2\pi y \sqrt{(3-3t^2)^2 + (6t)^2}\, dt$$

$$= \int_0^1 2\pi \cdot 3t^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\, dt$$

$$= \int_0^1 2\pi \cdot 3t^2 \sqrt{(3+3t^2)^2}\, dt$$

$$= \int_0^1 2\pi \cdot 3t^2 (3 + 3t^2)\, dt = 18\pi \int_0^1 (t^2 + t^4)\, dt$$

$$= 18\pi \cdot \left[\frac{t^3}{3} + \frac{t^5}{5}\right]\Bigg|_0^1 = \overset{6}{\cancel{18}}\pi \cdot \frac{8}{\underset{5}{\cancel{15}}}$$

$$= \boxed{\frac{48\pi}{5}}$$

Jul 9-9:38 AM

Intro. to Polar Coordinates

Rectangular (x, y)

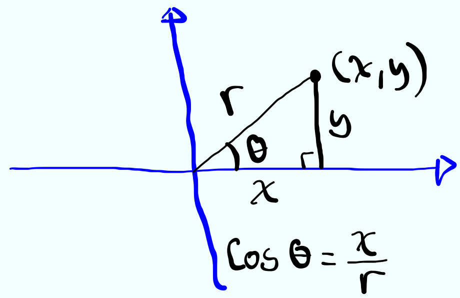
Polar coordinates (r, θ)

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{y}{x} = \tan\theta$

$x^2 + y^2 = r^2$



$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

Jul 9-9:48 AM

Graph $\frac{6}{2\cos\theta + 3\sin\theta} = r$

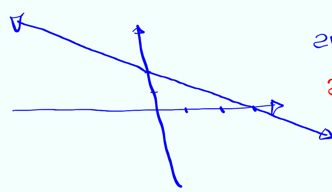
Polar equation

Cross-Multiply

$2r\cos\theta + 3r\sin\theta = 6$

$2x + 3y = 6$

x	y
0	2
3	0



Graph $r = 4\sin\theta$

multiply by r

$r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$

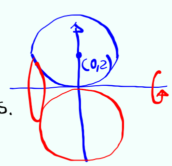
$x^2 + y^2 - 4y + 4 = 4$

$x^2 + (y-2)^2 = 2^2$

Circle

Center $(0,2)$

Radius 2



Find the volume if we rotate this enclosed region by x-axis.

Theorem of Pappus

$V =$ Area of the region $\cdot$ distance traveled by its centroid about axis of revolution

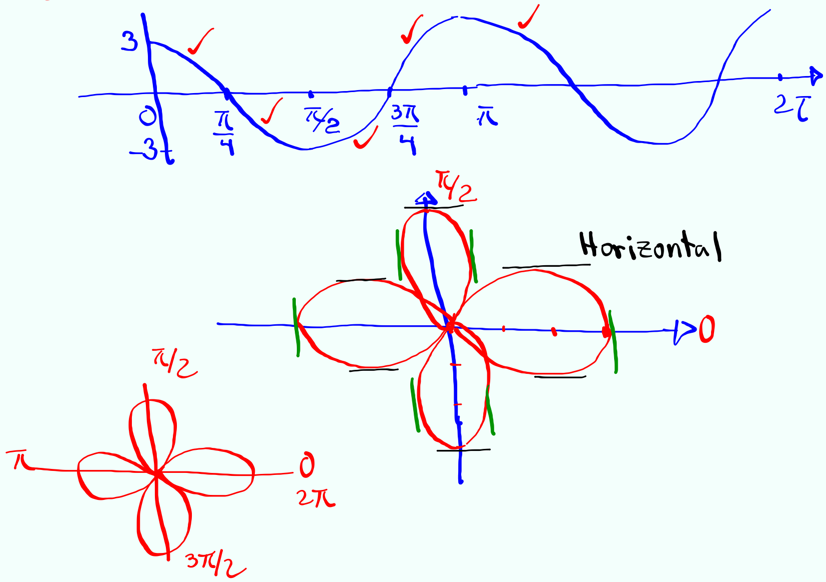
$V = 4\pi \cdot 2\pi\bar{y}$

$= 4\pi \cdot 2\pi \cdot 2 = 16\pi^2$

Jul 9-9:52 AM

Graph $r = 3\cos 2\theta$ Hold!

$y = 3\cos 2x$



Jul 9-10:03 AM

What was $\frac{dy}{dx}$?

Parametric Eqns $\quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Polar Eqns $\quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]}$

$$= \frac{\frac{dr}{d\theta}\cdot\sin\theta + r\cdot\cos\theta}{\frac{dr}{d\theta}\cdot\cos\theta - r\cdot\sin\theta}$$

$x = r\cos\theta$

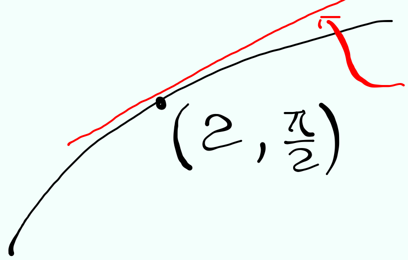
Recall $r = f(\theta)$

$y = r\sin\theta$

Jul 9-10:12 AM

Given $r = 1 + \sin\theta$

Find slope of the tan. line at $\theta = \frac{\pi}{2}$



$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}}$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\cdot\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cancel{\cos\theta}^{\,0}\cdot\sin\theta + (1+\sin\theta)\cdot\cancel{\cos\theta}^{\,0}}{\cancel{\cos\theta\cdot\cos\theta}^{\,0} - (1+\cancel{\sin\theta}^{\,1})\cancel{\sin\theta}^{\,1}}$$

$$= \frac{0}{-2} = 0$$

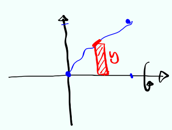
Jul 9-10:17 AM

$x = t^3$

$y = t^2$

$0 \le t \le 1$

Find the surface area by rotating the given graph by x-axis.



$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 2\pi t^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^1 2\pi t^2 \sqrt{\underbrace{(3t^2)^2 + (2t)^2}_{9t^4+4t^2}}\,dt$$

$$= 2\pi \int_0^1 t^2 \sqrt{9t^4 + 4t^2}\,dt = 2\pi \int_0^1 t^2 \cdot t\sqrt{9t^2+4}\,dt$$

$u = 9t^2 + 4$

$du = 18t\,dt$

$\frac{du}{18} = t\,dt$

$$= 2\pi \int_4^{13} \frac{u-4}{9}\cdot\sqrt{u}\,\frac{du}{18}$$

$$= \frac{\pi}{81} \int_4^{13} (u\sqrt{u} - 4\sqrt{u})\,du$$

$$= \frac{\pi}{81}\left[\frac{u^{5/2}}{5/2} - \frac{4u^{3/2}}{3/2}\right]\Bigg|_4^{13}$$

$$= \frac{\pi}{81}\left[\frac{2}{5}u^2\sqrt{u} - \frac{8}{3}u\sqrt{u}\right]\Bigg|_4^{13}$$

$$\boxed{\frac{2\pi}{1215}\left(247\sqrt{13} + 64\right)}$$

$$= \frac{\pi}{81}\left[\frac{2}{5}\cdot 13^2\sqrt{13} - \frac{8}{3}13\sqrt{13} - \frac{2}{5}\cdot 4^2\sqrt{4} + \frac{8}{3}\cdot 4\sqrt{4}\right]$$

$$= \frac{\pi}{81}\left[\frac{338\sqrt{13}}{5} - \frac{104\sqrt{13}}{3} - \frac{64}{5} + \frac{64}{3}\right]$$

$$= \frac{\pi}{81}\left[\frac{1014\sqrt{13} - 520\sqrt{13} - 192 + 320}{15}\right] = \frac{\pi}{1215}\left[494\sqrt{13} + 128\right]$$

$$= \frac{\pi}{1215}\left[2(247\sqrt{13} + 64)\right]$$

$$= \boxed{\frac{2\pi}{1215}\left[247\sqrt{13} + 64\right]}$$

Jul 8-10:56 AM